

# Two-fluid dynamics for a Bose-Einstein condensate out of local equilibrium with the non-condensate

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We extend our recent work on the two-fluid hydrodynamics of a Bose-condensed gas by including collisions involving both condensate and non-condensate atoms. These collisions are essential for establishing a state of local thermodynamic equilibrium between the condensate and non-condensate. Our theory is more general than the usual Landau two-fluid theory, to which it reduces in the appropriate limit, in that it allows one to describe situations in which a state of complete local equilibrium between the two components has not been reached. The exchange of atoms between the condensate and non-condensate is associated with a new relaxational mode of the gas.

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Recently the authors have given a microscopic derivation of the coupled two-fluid hydrodynamic equations for a trapped Bose-condensed gas [1]. In the present Letter, we report the results of a major extension of the ZGN work which takes into account the effects of collisions between the condensate and non-condensate atoms. These new equations of motion (which we shall call ZGN') allow us to discuss the dynamics when the non-condensate atoms are in local thermal equilibrium with each other (due to collisions between the excited atoms) but are not yet in complete equilibrium with the Bose condensate order parameter. This results in the appearance of a new relaxational collective mode related to the transfer of atoms between the condensate and non-condensate. In order to bring out the new physics in a clear fashion, these equations are solved for a uniform gas. A more complete derivation and discussion is given in Ref. [2].

The non-condensate atoms are described by the distribution function  $f(\mathbf{r}, \mathbf{p}, t)$ , which obeys the kinetic equation (we take  $\hbar = 1$  throughout)

$$\frac{\partial f}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla f - \nabla U \cdot \nabla_{\mathbf{p}} f = C_{12}[f] + C_{22}[f], \quad (1)$$

where the effective potential  $U(\mathbf{r}, t) \equiv U_{\text{ext}}(\mathbf{r}) + 2g[n_c(\mathbf{r}, t) + \tilde{n}(\mathbf{r}, t)]$  involves the self-consistent Hartree-Fock (HF) mean field. As usual, we treat the interactions in the  $s$ -wave approximation with  $g = 4\pi a/m$ . Here  $n_c(\mathbf{r}, t)$  is the condensate density and  $\tilde{n}(\mathbf{r}, t)$  is the non-condensate density given by

$$\tilde{n}(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi)^3} f(\mathbf{r}, \mathbf{p}, t). \quad (2)$$

A kinetic equation essentially equivalent to (1) in the case of a uniform system was given in Refs. [3,4].

The two collision terms in (1) are given by

$$C_{22}[f] \equiv 4\pi g^2 \int \frac{d\mathbf{p}_2}{(2\pi)^3} \int \frac{d\mathbf{p}_3}{(2\pi)^3} \int d\mathbf{p}_4 \times \delta(\mathbf{p} + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta(\tilde{\varepsilon}_p + \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3} - \tilde{\varepsilon}_{p_4}) \times [(1+f)(1+f_2)f_3f_4 - f f_2(1+f_3)(1+f_4)], \quad (3)$$

$$C_{12}[f] \equiv 4\pi g^2 n_c \int \frac{d\mathbf{p}_1}{(2\pi)^3} \int d\mathbf{p}_2 \int d\mathbf{p}_3 \times \delta(m\mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \delta(\varepsilon_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3}) \times [\delta(\mathbf{p} - \mathbf{p}_1) - \delta(\mathbf{p} - \mathbf{p}_2) - \delta(\mathbf{p} - \mathbf{p}_3)] \times [(1+f_1)f_2f_3 - f_1(1+f_2)(1+f_3)], \quad (4)$$

with  $f \equiv f(\mathbf{r}, \mathbf{p}, t)$ ,  $f_i \equiv f(\mathbf{r}, \mathbf{p}_i, t)$ . Eq. (4) takes into account the fact that a condensate atom locally has energy  $\varepsilon_c(\mathbf{r}, t) = \mu_c(\mathbf{r}, t) + \frac{1}{2}mv_c^2(\mathbf{r}, t)$  and momentum  $m\mathbf{v}_c$ , where the condensate chemical potential  $\mu_c$  and velocity  $\mathbf{v}_c$  will be defined in the next paragraph. On the other hand, a non-condensate atom locally has the HF energy  $\tilde{\varepsilon}_p(\mathbf{r}, t) = \frac{p^2}{2m} + U(\mathbf{r}, t)$ . This particle-like dispersion relation means that our analysis is limited to finite temperatures. It follows from this excitation spectrum that in the Landau limit (see later) of our microscopic model, the condensate density is equal to the superfluid density and the non-condensate density is the normal fluid density.

To complete our microscopic model, we need to have an equation of motion for the complex condensate order parameter  $\Phi(\mathbf{r}, t) \equiv \sqrt{n_c(\mathbf{r}, t)}e^{i\theta(\mathbf{r}, t)}$ . This equation can be rewritten in terms of  $n_c(\mathbf{r}, t)$  and the condensate velocity  $\mathbf{v}_c = \nabla\theta(\mathbf{r}, t)/m$  as

$$\frac{\partial n_c}{\partial t} + \nabla \cdot (n_c \mathbf{v}_c) = -\Gamma_{12}[f], \quad (5a)$$

$$m \left( \frac{\partial}{\partial t} + \mathbf{v}_c \cdot \nabla \right) \mathbf{v}_c = -\nabla \mu_c, \quad (5b)$$

where the condensate chemical potential (in the Thomas-Fermi approximation) is given by

$$\mu_c(\mathbf{r}, t) = U_{\text{ext}}(\mathbf{r}) + g[n_c(\mathbf{r}, t) + 2\tilde{n}(\mathbf{r}, t)]. \quad (6)$$

The “source” term  $\Gamma_{12}[f]$  in (5a) is defined in terms of the  $C_{12}$  collision term in (4) as

$$\Gamma_{12}[f] \equiv \int \frac{d\mathbf{p}}{(2\pi)^3} C_{12}[f(\mathbf{r}, \mathbf{p}, t)]. \quad (7)$$

We observe that  $C_{12}$  collisions do not conserve the number of atoms in the condensate.

The detailed derivation [2] of Eqs. (1-7) is based on a field-theoretic formulation of an interacting Bose fluid with a Bose broken symmetry. Closely related work can be found in Refs. [5] and [6]. The quantum field operators  $\hat{\psi}(\mathbf{r})$  are split into a condensate ( $\Phi \equiv \langle \hat{\psi} \rangle$ ) and non-condensate ( $\tilde{\psi}$ ) part. The key approximation made in obtaining (1) and (5) is the neglect of the anomalous pair correlations such as  $\langle \tilde{\psi}(\mathbf{r})\tilde{\psi}(\mathbf{r}) \rangle$ , which we shall refer to as the Popov approximation. In [1], the  $C_{12}$  collisions (involving one condensate atom) were not included and thus the source term  $\Gamma_{12}$  in (5a) was not present. In the extended set of ZGN’ hydrodynamic equations which follow from (1) and (5),  $\Gamma_{12}$  plays a crucial role. Above the Bose-Einstein transition temperature ( $T_{\text{BEC}}$ ), where the Bose condensate order parameter vanishes, the kinetic equation (1) only involves the Uehling-Uhlenbeck collision term  $C_{22}$  given by (3). This kinetic equation has been discussed extensively in recent years [7].

In this Letter, we restrict ourselves to the region in which  $C_{22}$  collisions are sufficiently rapid to justify the assumption that the excited-atom distribution function is approximately described by the local equilibrium Bose distribution

$$\tilde{f}(\mathbf{r}, \mathbf{p}, t) = \frac{1}{e^{\beta[\frac{1}{2m}(\mathbf{p}-m\mathbf{v}_n)^2 + U - \tilde{\mu}]} - 1}. \quad (8)$$

Here, the temperature parameter  $\beta$ , normal fluid velocity  $\mathbf{v}_n$ , chemical potential  $\tilde{\mu}$ , and mean field  $U$  are all functions of  $\mathbf{r}$  and  $t$ . It is important to appreciate that the local non-condensate chemical potential  $\tilde{\mu}$  which appears in (8) is distinct from the local condensate chemical potential  $\mu_c$ , as defined in (6). One may immediately verify that  $\tilde{f}$  satisfies  $C_{22}[\tilde{f}] = 0$ , independent of the value of  $\tilde{\mu}$ . In contrast, one finds from (4) that, in general,  $C_{12}[\tilde{f}] \neq 0$ . This means that even if the excited atoms are in dynamic local equilibrium described by (8), the source term  $\Gamma_{12}[f]$  in (5) will, in general, be finite. More specifically, we have (see also Ref. [5])

$$\Gamma_{12}[\tilde{f}] = - \left\{ 1 - e^{-\beta[\tilde{\mu} - \frac{1}{2}m(\mathbf{v}_n - \mathbf{v}_c)^2 - \mu_c]} \right\} \frac{n_c}{\tau_{12}}, \quad (9)$$

where we have introduced a collision time associated with the  $C_{12}$  term in (4),

$$\frac{1}{\tau_{12}} \equiv 4\pi g^2 \int \frac{d\mathbf{p}_1}{(2\pi)^3} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \int d\mathbf{p}_3 (1 + \tilde{f}_1)\tilde{f}_2\tilde{f}_3 \times \delta(m\mathbf{v}_c + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3) \delta(\varepsilon_c + \tilde{\varepsilon}_{p_1} - \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3}). \quad (10)$$

We note that  $\Gamma_{12}[\tilde{f}]$  in (9) vanishes when  $\tilde{\mu} = \mu_c + \frac{1}{2}m(\mathbf{v}_n - \mathbf{v}_c)^2$ . However, as we shall see, a state of complete local equilibrium cannot be treated simply by setting  $\Gamma_{12} = 0$ , which was the implicit assumption made in deriving the Landau two-fluid equations in earlier work [1,4].

With the assumption  $f \simeq \tilde{f}$ , one can derive hydrodynamic equations for the non-condensate by taking moments of (1) in the usual way. These are the analogue of the condensate equations in (5). Linearizing around the static thermal equilibrium solutions, these hydrodynamic equations are given by

$$\frac{\partial \delta \tilde{n}}{\partial t} = -\nabla \cdot (\tilde{n}_0 \delta \mathbf{v}_n) + \delta \Gamma_{12}, \quad (11a)$$

$$m\tilde{n}_0 \frac{\partial \delta \mathbf{v}_n}{\partial t} = -\nabla \delta \tilde{P} - \delta \tilde{n} \nabla U_0 - 2g\tilde{n}_0 \nabla (\delta \tilde{n} + \delta n_c), \quad (11b)$$

$$\frac{\partial \delta \tilde{P}}{\partial t} = -\frac{5}{3} \tilde{P}_0 \nabla \cdot \delta \mathbf{v}_n - \delta \mathbf{v}_n \cdot \nabla \tilde{P}_0 - \frac{2}{3} g n_{c0} \delta \Gamma_{12}. \quad (11c)$$

One also has

$$\tilde{n}(\mathbf{r}, t) = \int \frac{d\mathbf{p}}{(2\pi)^3} \tilde{f}(\mathbf{r}, \mathbf{p}, t) = \frac{1}{\Lambda^3} g_{3/2}(z), \quad (12)$$

$$\begin{aligned} \tilde{P}(\mathbf{r}, t) &= \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{p^2}{3m} \tilde{f}(\mathbf{r}, \mathbf{p}, t) \Big|_{\mathbf{v}_n=0} \\ &= \frac{1}{\beta \Lambda^3} g_{5/2}(z), \end{aligned} \quad (13)$$

where  $z \equiv e^{\beta(\tilde{\mu} - U)}$  is the local fugacity,  $\Lambda \equiv \sqrt{2\pi/mk_B T}$  is the local thermal de Broglie wavelength and  $g_n(z) \equiv \sum_{l=1}^{\infty} z^l / l^n$ . In static equilibrium (denoted by 0), we of course have  $\mathbf{v}_{n0} = \mathbf{v}_{c0} = 0$  and  $\mu_{c0} = \tilde{\mu}_0$ . Thus it follows that  $\Gamma_{12}[\tilde{f}^0] = 0$ . The analogous linearized condensate equations of motion are

$$\frac{\partial \delta n_c}{\partial t} = -\nabla \cdot (n_{c0} \delta \mathbf{v}_c) - \delta \Gamma_{12}, \quad (14a)$$

$$m \frac{\partial \delta \mathbf{v}_c}{\partial t} = -g \nabla (\delta n_c + 2\delta \tilde{n}). \quad (14b)$$

Finally, the source term  $\delta \Gamma_{12}$  in these equations can be expressed in terms of the fluctuation in the chemical potential difference  $\mu_{\text{diff}} \equiv \tilde{\mu} - \mu_c$ ,

$$\delta \Gamma_{12} = -\frac{\beta_0 n_{c0}}{\tau_{12}^0} \delta \mu_{\text{diff}}, \quad (15)$$

where  $\tau_{12}^0$  is the equilibrium collision time obtained from (10) with  $\mathbf{v}_c = 0$ ,  $\varepsilon_c = \mu_{c0}$  and  $\tilde{f}$  equal to the absolute

equilibrium Bose distribution. We note that adding (11a) and (14a) gives the usual continuity equation for the total density.

We now turn to a discussion of our linearized hydrodynamic equations given by (11)-(15) for a *uniform* Bose-condensed gas ( $U_{\text{ext}}(\mathbf{r}) = 0$ ). This means  $\tilde{n}_0$ ,  $n_{c0}$  and  $\tilde{P}_0$  are independent of position. By straightforward calculations [2], one can then reduce our two-fluid equations to three coupled equations of motion for the three variables  $\delta\mathbf{v}_c$ ,  $\delta\mathbf{v}_n$  and  $\delta\mu_{\text{diff}}$ :

$$m \frac{\partial^2 \delta\mathbf{v}_c}{\partial t^2} = gn_{c0} \nabla(\nabla \cdot \delta\mathbf{v}_c) + 2g\tilde{n}_0 \nabla(\nabla \cdot \delta\mathbf{v}_n) + \frac{\beta_0 gn_{c0}}{\tau_{12}^0} \nabla \delta\mu_{\text{diff}}, \quad (16a)$$

$$m \frac{\partial^2 \delta\mathbf{v}_n}{\partial t^2} = \left( \frac{5\tilde{P}_0}{3\tilde{n}_0} + 2g\tilde{n}_0 \right) \nabla(\nabla \cdot \delta\mathbf{v}_n) + 2gn_{c0} \nabla(\nabla \cdot \delta\mathbf{v}_c) - \frac{2n_{c0}}{3\tilde{n}_0} \frac{\beta_0 gn_{c0}}{\tau_{12}^0} \nabla \delta\mu_{\text{diff}}, \quad (16b)$$

$$\frac{\partial \delta\mu_{\text{diff}}}{\partial t} = gn_{c0} \left( \frac{2}{3} \nabla \cdot \delta\mathbf{v}_n - \nabla \cdot \delta\mathbf{v}_s \right) - \frac{\delta\mu_{\text{diff}}}{\tau_\mu}. \quad (16c)$$

Here the relaxation time for the chemical potential difference ( $\mu_{\text{diff}}$ ) due to  $C_{12}$  collisions between the condensate and non-condensate atoms is given by the expression [2]

$$\frac{1}{\tau_\mu} \equiv \frac{\beta_0 gn_{c0}}{\tau_{12}^0} \left( \frac{\frac{5}{2}\tilde{P}_0 + 2g\tilde{n}_0 n_{c0} + \frac{2}{3}\tilde{\gamma}_0 gn_{c0}^2}{\frac{5}{2}\tilde{\gamma}_0 \tilde{P}_0 - \frac{3}{2}g\tilde{n}_0^2} - 1 \right), \quad (17)$$

where we have introduced the dimensionless function  $\tilde{\gamma}_0 \equiv (g\beta_0/\Lambda_0^3)g_{1/2}(z_0)$ . If we simply omit the terms involving  $\delta\mu_{\text{diff}}$  in (16a) and (16b), we are left with the two coupled ZGN equations for  $\delta\mathbf{v}_n$  and  $\delta\mathbf{v}_c$  given in Ref. [8]. We see that our new generalized ZGN' equations give rise to a coupling between  $\delta\mathbf{v}_n$  and  $\delta\mathbf{v}_c$  and the local variable  $\delta\mu_{\text{diff}}$ , which describes the relative fluctuation in the chemical potentials of the two components. This is the most important new result in the present Letter.

It is convenient to solve the three coupled equations in (16) by introducing velocity potentials  $\delta\mathbf{v}_c \equiv \nabla\phi_c$  and  $\delta\mathbf{v}_n \equiv \nabla\phi_n$ , and looking for plane wave solutions. We obtain from (16c)

$$(1 - i\omega\tau_\mu)\delta\mu_{\text{diff}} = gn_{c0}\tau_\mu \left( \phi_c - \frac{2}{3}\phi_n \right) k^2. \quad (18)$$

Inserting this expression into (16a) and (16b) gives two coupled equations for  $\phi_n$  and  $\phi_c$ ,

$$\begin{aligned} m\omega^2\phi_c &= gn_{c0} \left[ 1 - \frac{\beta_0 gn_{c0}\tau_\mu}{\tau_{12}^0(1 - i\omega\tau_\mu)} \right] k^2\phi_c \\ &\quad + 2g\tilde{n}_0 \left[ 1 + \frac{\beta_0 gn_{c0}\tau_\mu}{3\tau_{12}^0(1 - i\omega\tau_\mu)} \frac{n_{c0}}{\tilde{n}_0} \right] k^2\phi_n, \quad (19a) \\ m\omega^2\phi_n &= 2gn_{c0} \left[ 1 + \frac{\beta_0 gn_{c0}\tau_\mu}{3\tau_{12}^0(1 - i\omega\tau_\mu)} \frac{n_{c0}}{\tilde{n}_0} \right] k^2\phi_c \end{aligned}$$

$$+ \left[ \frac{5\tilde{P}_0}{3\tilde{n}_0} + 2g\tilde{n}_0 - \frac{4\beta_0(gn_{c0})^2\tau_\mu}{9\tau_{12}^0(1 - i\omega\tau_\mu)} \frac{n_{c0}}{\tilde{n}_0} \right] k^2\phi_n. \quad (19b)$$

Taking the limit  $\omega\tau_\mu \gg 1$ , one finds that  $\delta\mu_{\text{diff}}$  is decoupled from the velocity potentials  $\phi_{c,n}$  and we recover the ZGN results [8]. As expected, in the extreme limit  $\omega\tau_\mu \gg 1$ , the effect of  $C_{12}$  collisions are negligible and one can simply omit  $\Gamma_{12}$ .

In the opposite limit  $\omega\tau_\mu \rightarrow 0$ , the equations in (19) yield two phonon-like solutions,  $\omega_{1,2} = u_{1,2}k$ , where the velocities are given by the roots of the equation  $u^4 - Au^2 + B = 0$ . It can be shown that the coefficients  $A$  and  $B$  are in *exact* agreement with the analogous coefficients obtained from the usual Landau two-fluid equations [8,9]. The latter theory uses quite different thermodynamic variables from those used in the present formulation, and the explicit proof of this equivalence requires a lengthy (but straightforward) calculation. It also turns out that the first and second sound velocities ( $u_1$  and  $u_2$ ) given by these results (valid for  $\omega\tau_\mu \ll 1$ ) are numerically very close to the velocities given by the ZGN approximation (valid for  $\omega\tau_\mu \gg 1$ ). The small differences involve terms of order  $g^2$  and thus were not picked up in the comparison given in Ref. [8].

The interesting feature of the linearized ZGN' equations in (18) and (19) is the existence of a new mode, associated with the condensate and non-condensate being out of equilibrium ( $\delta\mu_c \neq \delta\mu$ ). To a good approximation (and exact in the  $k \rightarrow 0$  limit), it corresponds to a mode in which  $\delta\mathbf{v}_n = \delta\mathbf{v}_c = 0$ , with a frequency given by  $\omega = -i/\tau_\mu$ . In the ZGN limit ( $C_{12} = 0$ ), this reduces to a zero frequency mode. In the Landau limit ( $C_{12}$  large), it is a heavily damped relaxational mode. According to (19), this equilibration process also gives rise to a damping of second sound whose magnitude relative to the mode frequency is peaked at  $\omega\tau_\mu \simeq 1$ .

In Fig. 1, we plot the reciprocals of various relaxation times involved in our linearized ZGN' equations, as a function of temperature. We note that  $\tau_{12}^0$  goes to zero at  $T_{\text{BEC}}$  and is much smaller than the mean collision time expected for a Maxwell-Boltzmann gas. The extra factors multiplying  $1/\tau_{12}^0$  in (17) ensure that  $1/\tau_\mu$  starts to decrease as we approach  $T_{\text{BEC}}$  from below. This is the expected "critical slowing down" seen in all second order phase transitions involving an order parameter. Our HF mean-field approximation is inadequate in the critical region close to  $T_{\text{BEC}}$ , and leads to a spurious finite limiting value of  $n_{c0}$  at  $T_{\text{BEC}}$  [8]. This removes the divergence in  $1/\tau_{12}^0$ , which is also why  $\tau_\mu$  in Fig. 1 is finite at  $T_{\text{BEC}}$ . If we simply put  $n_{c0} = 0$  in our calculations, we would find that both  $\tau_\mu$  and  $1/\tau_{12}^0$  would diverge at  $T_{\text{BEC}}$ .

Our discussion in the present Letter is based on the assumption that  $C_{22}$  collisions produce the local equilibrium distribution  $\tilde{f}$  in (8). As a result, our hydrodynamic equations (11) do not explicitly depend on a

relaxation time associated with  $C_{22}$ . This could be included using the standard Chapman-Enskog approach to deal with the deviation of  $f$  from the local equilibrium function  $\tilde{f}$  [4,10]. However, an estimate of the collision time associated with  $C_{22}$  is given by the scattering out term in (3),

$$\frac{1}{\tau_{22}^0} = \frac{4\pi g^2}{\tilde{n}_0} \int \frac{d\mathbf{p}_1}{(2\pi)^3} \int \frac{d\mathbf{p}_2}{(2\pi)^3} \int \frac{d\mathbf{p}_3}{(2\pi)^3} \int d\mathbf{p}_4 \times \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \delta(\tilde{\varepsilon}_{p_1} + \tilde{\varepsilon}_{p_2} - \tilde{\varepsilon}_{p_3} - \tilde{\varepsilon}_{p_4}) \times \tilde{f}_1^0 \tilde{f}_2^0 (1 + \tilde{f}_3^0)(1 + \tilde{f}_4^0). \quad (20)$$

This collision time is plotted in Fig. 1, both above and below  $T_{\text{BEC}}$ . The fact that  $\tau_{22}^0 \ll \tau_\mu$  at temperatures  $T \gtrsim 0.8 T_{\text{BEC}}$  is very important, since it allows for the possibility that the non-condensate atoms are in local equilibrium with each other but not with the condensate. Above the transition, the value for  $\tau_{22}^0$  we obtain is in close agreement with the collision time obtained in Ref. [7].

These results for the relaxation times are quite interesting in their own right. The divergence at  $T_{\text{BEC}}$  (see above remarks) is a consequence of the Bose distribution function being used for  $\tilde{f}_i^0$  in (10) and (20). If a Maxwell-Boltzmann distribution function were used, the “divergence” shown in Fig. 1 is removed (the importance of calculating collision times using the correct Bose distribution has also been noted in Ref. [5]). Moreover, we see that one should not use the classical gas approximation for the collision times when determining the cross-over from a collisionless (or mean-field) to hydrodynamic regimes. The results in Fig. 1 imply that the hydrodynamic domain is much easier to reach at finite temperatures than might have been expected, since the collision time can be much *smaller* than the analogous classical gas collision time.

In summary, starting from a microscopic model, we find that the dynamics of a Bose-condensed gas at finite temperatures can be divided into *three* distinct regimes: (1) The collisionless regime in which no collision terms are included in the kinetic equation ( $C_{12} = C_{22} = 0$ ). (2) An intermediate regime in which  $C_{22}$  collisions between the excited atoms establish local thermal equilibrium ( $\omega\tau_{22}^0 \ll 1$ ) but the  $C_{12}$  collisions do not keep the condensate in equilibrium with the non-condensate [3,11]. The relaxation time  $\tau_\mu$  for this equilibration is found to be much larger than the collision time  $\tau_{22}^0$  for reaching local equilibrium in the non-condensate (see Fig. 1). (3) Complete local equilibrium of the condensate and non-condensate, which arises in the limit  $\omega\tau_\mu \ll 1$ . This is the regime conventionally described by the Landau two-fluid equations [8,9]. As stated earlier, the ZGN’ equations exactly reproduce the results of the Landau equations, even though the local dynamical variables used are quite different in the two approaches.

In this Letter, we have only discussed the normal modes of our linearized equations in a uniform Bose gas,

but similar conclusions are obtained for trapped gases [2]. Our general equations can also be used to discuss the growth and decay of atomic condensates [5,12], taking the dynamics of the non-condensate into account.

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FIG. 1. Various collision and relaxation times for a uniform Bose gas as a function of temperature, for  $gn = 0.1k_B T_{\text{BEC}}$ . The values are normalized to the classical collision time  $\tau_{\text{cl}}^{-1} = n\sigma(16k_B T/\pi m)^{1/2}$  at  $T = T_{\text{BEC}}$ , obtained from (20) using a Maxwell-Boltzmann distribution for  $f$ . Here,  $\sigma = 8\pi a^2$  is the atomic cross-section. The temperature scale changes abruptly above  $T_{\text{BEC}}$ . The present calculations are not valid in the very low temperature region.

**Figure 1**

